

# Brown representability for exterior cohomology and cohomology with compact supports

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## Abstract

It is well known that cohomology with compact supports is not a homotopy invariant but only a proper homotopy one. However, as the proper category lacks of general categorical properties, a Brown representability theorem type does not seem reachable. However, by proving such a theorem for the so called exterior cohomology in the complete and cocomplete exterior category, we show that the  $n$ -th cohomology with compact supports of a given countable, locally finite, finite dimensional relative CW-complex  $(X, \mathbb{R}_+)$  is naturally identified with the set  $[X, K_n]^{\mathbb{R}_+}$  of exterior based homotopy classes from a “classifying space”  $K_n$ . We also show that this space has the exterior homotopy type of the exterior Eilenberg-MacLane space for Brown-Grossman homotopy groups of type  $(R^\infty, n)$ ,  $R$  being the fixed coefficient ring.

## Introduction

Proper homotopy theory is designed to study non compact topological spaces modulo deformations which respect the behaviour of these spaces at infinity. In this paper we are concerned with the classification of cohomology invariants of proper homotopy. Among them, classical cohomology with compact supports and locally finite cohomology [20, §3] are specially well adapted functors to study locally compact spaces up to proper homotopy.

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In classical homotopy theory, the representation, in the Brown sense, of cohomology functors in terms of homotopy classes of maps coming from a universal space was the starting point of many important developments. Proper homotopy theory would also very much benefit from this classification. However, due to the lack of categorical properties of the proper category  $\mathbf{P}$ , the behaviour of classical homotopy invariants are not easy, when possible, to translate to the proper setting. This applies to the mentioned problem: first of all, the above cohomology theories are not homotopy invariants but only proper homotopy invariants, so classical Brown representability cannot be attained within this category. Indeed, none of the various representability theorems available applies as  $\mathbf{P}$  is not closed even for finite limits or colimits. Recall that the proper category is only a cofibration category [1] and that the most general forms of Brown representability theorem, see for instance [21, 22, 23], requires the category to which it applies to be, among other conditions, cocomplete.

We overcome these obstructions via the following procedure: consider the full embedding of  $\mathbf{P}$  into the so called *exterior category*  $\mathbf{E}$  (see next section for precise definition and details), a complete and cocomplete category in which exterior homotopy theory can be developed [16]. We can also consider the *exterior cohomology* in  $\mathbf{E}$ , an extension of compactly supported cohomology to the exterior category. This already appears in [25, §2] under a different approach, and it has recently been used to obtain interesting results in proper homotopy theory [7, 8, 18]. Finally, we show that the exterior cohomology satisfies all the necessary properties to be classified in the Brown sense.

As a result of this procedure we prove the following representability theorem for the exterior cohomology of the, so called, exterior CW-complexes, (see next sections for precise definitions and notation).

**Theorem 0.1.** *For each  $n \geq 0$ , there exist an  $e$ -path connected, exterior CW-complex  $K_n \in \mathbf{HoCW}^{\mathbb{R}_+}$ , unique up to based exterior homotopy, and a universal element  $u \in H_{\mathcal{E}}^n(K_n)$ , such that the natural natural transformation*

$$T_u : [-, K_n]^{\mathbb{R}_+} \longrightarrow H_{\mathcal{E}}^n(-), \quad T_u[f] = H_{\mathcal{E}}^n(f)(u),$$

*induces a bijection  $[X, K_n]^{\mathbb{R}_+} \cong H_{\mathcal{E}}^*(X)$  for any exterior CW-complex  $X \in \mathbf{HoCW}^{\mathbb{R}_+}$ .*

The translation of this result to the classical cohomology with compact supports on the proper category reads:

**Theorem 0.2.** *Let  $(X, \mathbb{R}_+)$  be a countable, locally finite, finite dimensional relative CW-complex, and let  $n \geq 0$ . There exist an  $e$ -path connected, exterior CW-complex  $K_n \in \mathbf{HoCW}^{\mathbb{R}_+}$  (not necessarily endowed with the cocompact externology!), unique up to based exterior homotopy, and a universal element  $u \in H_{\mathcal{E}}^n(K_n)$ , such that the map*

$$T_u : [X, K_n]^{\mathbb{R}_+} \xrightarrow{\cong} H_c^n(X), \quad T_u[f] = H_{\mathcal{E}}^n(f)(u),$$

*is a bijection.*

In other words, cohomology with compact supports of one ended spaces in the proper category are classified by homotopy classes of based exterior maps into a classifying space which generally lives outside the proper category.

We finish by describing, as in the classical case, the exterior homotopy type of the classifying space for the exterior cohomology as an Eilenberg-MacLane space for Brown-Grossman homotopy groups. See Theorem 3.4.

In the next section we present a brief summary of exterior homotopy theory. In Section 2 we set the main properties of the exterior cohomology by regarding it as the cohomology of the so called *Alexandroff construction* of the given exterior space. In Section 3 we prove Theorems 0.1, 0.2 and explicitly describe the exterior homotopy type of the classifying space for exterior cohomology.

## 1 Exterior homotopy theory

For a brief review of fundamental results on exterior homotopy theory we refer to [14, §1]. Here we simply recall the basic concepts and facts we will use.

A *proper* map is a continuous map  $f : X \rightarrow Y$  such that  $f^{-1}(K)$  is a compact subset of  $X$ , for every closed compact subset  $K$  of  $Y$ . We will denote by  $\mathbf{P}$  the category of spaces and proper maps. In proper homotopy theory the role of the base point is played by the half-line  $\mathbb{R}_+ = [0, \infty)$ .

An *exterior space*  $(X, \mathcal{E})$  consists of a topological space  $(X, \tau)$  together with a non empty family of *exterior sets*  $\mathcal{E} \subset \tau$ , called *externology* which is closed under finite intersections and, whenever  $U \supset E$ ,  $E \in \mathcal{E}$ ,  $U \in \tau$ , then  $U \in \mathcal{E}$ . We may think of  $\mathcal{E}$  as a neighborhood system at infinity. A map  $f : (X, \mathcal{E}) \rightarrow (X', \mathcal{E}')$  is *exterior* if it is continuous and  $f^{-1}(E) \in \mathcal{E}$ , for all  $E \in \mathcal{E}'$ . A subset  $A \subset X$  is called an *exterior neighborhood* if it contains an exterior set. The complement of an exterior set is called an exterior closed set or *e-closed*. The category of exterior spaces shall be denoted by  $\mathbf{E}$ .

For a given topological space  $X$  we can consider its *cocompact externology*  $\mathcal{E}_{cc}$  which is formed by the family of the complements of all closed-compact sets of  $X$ . The corresponding exterior space will be denoted by  $X_{cc}$ . The correspondence  $X \mapsto X_{cc}$  gives rise to a full embedding [16, Thm. 3.2]

$$(-)_{cc}: \mathbf{P} \hookrightarrow \mathbf{E}.$$

Furthermore, the category  $\mathbf{E}$  is complete and cocomplete [16, Thm. 3.3].

Let  $X$  and  $Y$  be an exterior and a topological space respectively. On the product space  $X \times Y$  consider the following externology: an open set  $E$  is exterior if for each  $y \in Y$  there exists an open neighborhood of  $y$ ,  $U_y$ , and an exterior open  $E_y$  such that  $E_y \times U_y \subset E$ . We denote by  $X \bar{\times} Y$  the resulting exterior space. If  $Y$  is compact, then  $E$  is an exterior open if and only if it is an open set and there exists  $G \in \mathcal{E}_X$  for which  $G \times Y \subset E$ . In particular, if  $\mathcal{E}_X = \mathcal{E}_{cc}^X$  and  $Y$  is compact, then  $X_{cc} \bar{\times} Y = (X \times Y)_{cc}$ . Hence, the *cylinder*

$$-\bar{\times} I: \mathbf{E} \rightarrow \mathbf{E},$$

together with the obvious natural transformations

$$\iota_0, \iota_1: id \rightarrow -\bar{\times} I, \quad \rho: -\bar{\times} I \rightarrow id,$$

provide a natural way to define *exterior homotopy* in  $\mathbf{E}$ . This functor restricts to the proper category

$$-\bar{\times} I = - \times I: \mathbf{P} \rightarrow \mathbf{P}.$$

It is worth mentioning that, unlike in the proper framework [1, Thm. 1.4], the exterior cylinder has a right adjoint [16, Thm. 3.5]

$$(-)^I: \mathbf{E} \rightarrow \mathbf{E}$$

which also leads, this time via paths, to the same notion of exterior homotopy.

Given exterior spaces  $X, Z$ , the mapping set  $Z^X$  of exterior maps is canonically endowed with the topology generated by

$$(K, U) = \{\alpha \in Z^X, \alpha(K) \subset U\} \text{ and } (L, E) = \{\alpha \in Z^X, \alpha(L) \subset E\}$$

where  $K$  is compact,  $U$  is open,  $E$  is exterior, and  $L$  is e-compact, that is,  $L \setminus F$  is compact for any exterior  $F$ . Given a Hausdorff, locally compact space  $X$  endowed with the cocompact externology, there is a natural bijection

$$Hom_E(X \bar{\times} Y, Z) \cong Hom_{Top}(Y, Z^X).$$

The right framework for pointed exterior homotopy is the category  $\mathbf{E}^{\mathbb{R}_+}$  of *based exterior spaces* or *exterior spaces under  $\mathbb{R}_+$*  in which  $\mathbb{R}_+$  is endowed with the cocompact externology. Its objects are pairs  $(X, \alpha)$ , where  $\alpha: \mathbb{R}_+ \rightarrow X$  is an exterior map (the *base ray*). Morphisms  $f: (X, \alpha) \rightarrow (Y, \beta)$  are exterior maps  $f: X \rightarrow Y$  for which  $f\alpha = \beta$ .

From now on  $\mathbb{N} \subset \mathbb{R}_+$  will always be endowed with the induced externology. Also, any compact space is considered with the topology as externology. Note that any  $(X, \alpha) \in \mathbf{E}^{\mathbb{R}_+}$  may, and will henceforth, be considered in the category  $\mathbf{E}^{\mathbb{N}}$  of exterior spaces under  $\mathbb{N}$  by composing  $\alpha$  with the exterior inclusion  $\mathbb{N} \hookrightarrow \mathbb{R}$ .

Given  $(X, \alpha) \in \mathbf{E}^{\mathbb{R}_+}$ , the *based exterior cylinder of  $X$* ,  $I^{\mathbb{R}_+}X$ , is defined by the pushout in  $\mathbf{E}$ :

$$\begin{array}{ccc} \mathbb{R}_+ \bar{\times} I & \xrightarrow{\rho} & \mathbb{R}_+ \\ \alpha \bar{\times} id \downarrow & & \downarrow \\ X \bar{\times} I & \longrightarrow & I^{\mathbb{R}_+}X \end{array}$$

The notion of *based exterior homotopy* is then naturally defined in the obvious way. This can be dually introduced by means of the *based exterior cocylinder of  $X$* .

We also need to recall the notion of well based exterior spaces. An exterior map  $j: A \rightarrow X$  is an *exterior cofibration* if it satisfies the Homotopy Extension Property (*HEP*) in  $\mathbf{E}$ . That is, if for any commutative diagram of exterior maps

$$\begin{array}{ccc} A & \xrightarrow{\iota_0} & A \bar{\times} I \\ j \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array} \quad \begin{array}{c} \nearrow j \bar{\times} id \\ \searrow \iota_0 \\ X \bar{\times} I \end{array}$$

the dotted arrow always exists. In the proper setting, if  $j$  is a proper map between Hausdorff and locally compact spaces, then  $j$  is a proper cofibration if and only if  $j_{cc}$  is an exterior cofibration. Recall that a proper cofibration is a proper map which satisfies the corresponding proper homotopy extension property.

A based exterior space  $(X, \alpha)$  is *well based* if  $\alpha$  is an exterior closed cofibration. We denote the corresponding full subcategory by  $\mathbf{E}_w^{\mathbb{R}_+} \subset \mathbf{E}^{\mathbb{R}_+}$ . We point out that  $\mathbf{E}_w^{\mathbb{R}_+}$  verifies all the axioms required for a closed model category, except for being closed for finite limits and colimits [13, Thm. 2.12].

Fibrations (resp. cofibrations) are exterior based maps which verify the Homotopy Lifting Property (resp. the Homotopy Extension Property), while weak equivalences are based exterior homotopy equivalences. Nevertheless, many of such limits, as pullbacks of fibrations and pushouts of cofibrations, can be constructed within  $\mathbf{E}_w^{\mathbb{R}_+}$  and thus, *exterior homotopy pullbacks and pushouts* are defined within this category.

Next, we recall the notion of *exterior CW-complex* we use. It is slightly more general than the original one introduced in [14], which include for instance, the classical CW-complexes (with topology as externology) and, in the proper case, the spherical objects under a tree of [2, §2], [3, IV] and [4, 1.3].

Given  $n \geq 0$ , we denote by  $\mathfrak{S}^k$  either the classical  $k$ -dimensional sphere  $S^k$  or the  $k$ -dimensional  $\mathbb{N}$ -sphere  $\mathbb{N} \bar{\times} S^k$ . Analogously  $\mathfrak{D}^k$  will ambiguously denote either the classical  $k$ -dimensional disk  $D^k$  or  $\mathbb{N} \bar{\times} D^k$ , the  $k$ -dimensional  $\mathbb{N}$ -disk. The inclusions  $\mathfrak{S}^{k-1} \hookrightarrow \mathfrak{D}^k$  are exterior cofibrations.

A *relative exterior CW-complex*  $(X, A)$  is an exterior space  $X$  together with a filtration of exterior subspaces

$$A = X^{-1} \subset X^0 \subset X^1 \subset \dots \subset X^n \subset \dots \subset X$$

for which  $X = \text{colim } X^n$ , and for each  $n \geq 0$ ,  $X^n$  is obtained from  $X^{n-1}$  as the exterior pushout

$$\begin{array}{ccc} \sqcup_{\gamma \in \Gamma} \mathfrak{S}_{\gamma}^{n-1} & \xrightarrow{\sqcup_{\gamma \in \Gamma} \varphi_{\gamma}} & X^{n-1} \\ \downarrow & & \downarrow \\ \sqcup_{\gamma \in \Gamma} \mathfrak{D}_{\gamma}^n & \xrightarrow{\sqcup_{\gamma \in \Gamma} \psi_{\gamma}} & X^n \end{array}$$

via the attaching maps  $\varphi_{\gamma}: \mathfrak{S}_{\gamma}^{n-1} \rightarrow X^{n-1}$ . The resulting maps  $\psi_{\gamma}$  are called *characteristic* and the exterior spaces  $\psi_{\gamma}(\mathfrak{D}_{\gamma}^n)$  are the *cells* of dimension  $n$  of  $X$ . If  $A = \emptyset$ , then  $X$  is simply called *exterior CW-complex*. On the other hand, if  $A = \mathbb{R}_+$ , then  $(X, \mathbb{R}_+)$  is called a *based exterior CW-complex*.

Note that  $(X, \mathbb{R}_+) \in \mathbf{E}_w^{\mathbb{R}_+}$  as the inclusion  $\mathbb{R}_+ \hookrightarrow X$  is a closed cofibration. We will denote by  $\mathbf{CW}^{\mathbb{R}_+}$ , (resp.  $\mathbf{CW}_f^{\mathbb{R}_+}$ ) the full subcategory of  $\mathbf{E}_w^{\mathbb{R}_+}$  formed by based exterior CW-complexes (resp. finite based exterior CW-complexes). We denote by  $\mathbf{HoCW}^{\mathbb{R}_+}$  (resp.  $\mathbf{HoCW}_f^{\mathbb{R}_+}$ ) its corresponding homotopy category with the same objects, and whose set of morphisms  $[X, Y]^{\mathbb{R}_+}$  between  $(X, \mathbb{R}_+)$  and  $(Y, \mathbb{R}_+)$ , or simply  $X$  and  $Y$  for simplicity, are based exterior homotopy classes of exterior maps. Remark that a finite

exterior CW-complex is in general a finite dimensional infinite classical CW-complex. Also, any classical CW-complex is an exterior CW-complex with its topology as externology. Other important class of exterior CW-complexes are constituted by the open differential manifolds and PL-manifolds as they admit a locally finite countable triangulation, which describes the exterior CW-structure [14, §2(ii)].

On the other hand, if  $(X, A)$  is any countable, locally finite, finite dimensional relative CW-complex in which  $A$  is a Hausdorff locally compact space, then it is not difficult to check that  $(X_{cc}, A_{cc})$  is a relative exterior CW-complex [18], [13, Prop. 2.3]. As we will use them later, we point out that there are exterior versions of the Whitehead and cellular approximation theorems for exterior CW-complexes [14, Thms. 12,13,14], [19, Thms. 4.1.19,4.1.26].

We finish this section by recalling how homotopy groups are introduced in the exterior setting. For it, consider  $\mathbb{N}$ -spheres  $\mathfrak{S}^k = \mathbb{N} \bar{\times} S^k$  and  $\mathbb{N}$ -disks  $\mathfrak{D}^{k+1} = \mathbb{N} \bar{\times} D^{k+1}$  as spaces in  $\mathbf{E}^{\mathbb{N}}$  via the natural exterior inclusions  $\eta: \mathbb{N} \hookrightarrow \mathfrak{S} \subset \mathfrak{D}$ ,  $\eta(n) = (n, *)$ . Given any  $(X, \alpha) \in \mathbf{E}^{\mathbb{R}+}$  (or more generally, in  $\mathbf{E}^{\mathbb{N}}$ ), and any  $k \geq 0$ , the  $k$ -th *Brown-Grossman exterior homotopy group* of  $(X, \alpha)$  [17, 2.3] is defined as

$$\pi_k^{\mathfrak{B}}(X, \alpha) = [(\mathfrak{S}^k, \eta), (X, \alpha)]^{\mathbb{N}}$$

where  $[-, -]^{\mathbb{N}}$  denotes the set of homotopy classes under  $\mathbb{N}$ . The group structure, for  $k \geq 1$ , is given by the natural bijection

$$[(\mathfrak{S}^k, \eta), (X, \alpha)]^{\mathbb{N}} \cong [(S^k, *), (X^{\mathbb{N}}, \alpha)]^*$$

with the ordinary  $k$ -th homotopy group of the pair  $(X^{\mathbb{N}}, \alpha)$  (see [16, Prop. 3.2] or [17, Rem. 2.4]).

A “continuous” way to see these homotopy groups is the following (see for instance [14, Rem. 7]). For each  $n \geq 1$  consider the based, finite, exterior CW-complex  $S_{\eta}^n$  obtained as the pushout

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\eta} & \mathfrak{S}^n \\ \downarrow & & \downarrow \\ \mathbb{R}_+ & \xrightarrow{\xi} & S_{\eta}^n. \end{array}$$

That is,  $S_{\eta}^n$  is the half real line, with an  $n$ -sphere attached to any integer number, and endowed with the cocompact externology. For any based exterior space  $(X, \alpha)$ , the universal property of the pushout provides a canonical

isomorphism

$$[(S_\eta^n, \xi), (X, \alpha)]^{\mathbb{R}_+} \cong [(\mathfrak{S}^n, \eta), (X, \alpha)]^{\mathbb{N}} = \pi_n^{\mathfrak{B}}(X, \alpha).$$

An exterior space  $X \in \mathbf{E}$  is *exterior  $n$ -connected* if it is  $n$ -connected in the classical sense, and  $\pi_k^{\mathfrak{B}}(X, \alpha) = \{0\}$ ,  $0 \leq k \leq n$ , for any exterior base sequence  $\alpha: \mathbb{N} \rightarrow X$ . The space  $X$  is  *$e$ -path connected* if it is exterior 0-connected.

An exterior map  $f: Y \rightarrow Z$  is an *exterior  $n$ -equivalence* if it is a classical  $n$ -equivalence and  $f_*: \pi_k^{\mathfrak{B}}(Y, \alpha) \rightarrow \pi_k^{\mathfrak{B}}(Z, f\alpha)$  is isomorphism for  $0 \leq k \leq n-1$  and surjective for  $k = n$ , for any exterior base sequence  $\alpha: \mathbb{N} \rightarrow Y$ .

## 2 Exterior cohomology

In this section, as stated in the introduction, we set the main properties of *exterior cohomology*, already introduced in [25, §2] under a different approach, and successfully used in [7, 8, 18]. From now on, and unless explicitly stated otherwise, we fix a continuous and additive classical cohomology theory, i.e., a contravariant functor  $H^*$  from the category of pairs of spaces to the category of non negatively graded abelian groups satisfying exactness, excision, homotopy invariance, continuity (tautness), and additivity. By continuity we mean the following: for every triad  $(X, A, B)$  in which  $(A, B)$  is a closed pair,  $H^*(A, B) = \varinjlim H^*(Y, Z)$ , with  $(Y, Z)$  closed neighborhood of  $(A, B)$  in  $X$ . On the other hand, additivity means that, for any family  $\{X_i\}_{i \in I}$ ,  $H^*(\coprod_{i \in I} X_i) \cong \prod_{i \in I} H^*(X_i)$ .

*Remark 2.1.* Our cohomology set of axioms may not be the most general one, see for instance [26], to which what follows can be applied. However, classical theories like Čech, or Alexander-Spanier cohomology, fit within our framework. Moreover, it is worth remarking that both of them coincide with singular cohomology when we restrict to *HLC spaces* [5, Chap. III]. Recall that a space  $X$  is *homologically locally connected* (HLC) if for every neighborhood  $U$  of a given point  $x \in X$  there is another neighborhood  $V \subset U$  of  $x$ , such that the morphism induced in reduced singular homology  $\tilde{H}_*(V) \rightarrow \tilde{H}_*(U)$  is trivial. In particular, CW-complexes, or more generally, locally contractible spaces are HLC and, for them, our main results hold even when considering singular cohomology theory.

**Definition 2.2.** Let  $(X, \mathcal{E})$  be an exterior space. Define the *exterior cohomology* of  $X$  as,

$$H_{\mathcal{E}}^*(X) = \varinjlim \{H^*(X, E), E \in \mathcal{E}\}.$$



This defines a contravariant functor

$$H_{\mathcal{E}}^*: \mathbf{E} \longrightarrow \mathbf{Sets}.$$

*Remark 2.3.* (i) Observe that this definition coincides in most cases with the one in [25, Thm. 3.2]. Indeed, a *family of supports*  $\phi$  of a space  $X$ , a concept which goes back to [5, I.6], is precisely formed by the complements of the elements of an externology  $\mathcal{E}$  of  $X$ , i.e.,  $\Phi = \{E^c, E \in \mathcal{E}\}$ . Moreover, it is straightforward to show that a *co- $\Phi$  set* of [25] is nothing but an exterior neighborhood in our terminology.

(ii) For the special case of singular cohomology, exterior cohomology was already introduced in [7] by considering, for each  $E \in \mathcal{E}$ , the complex  $C^*(X, E)$  of singular cochains of  $X$  which vanish in  $E$ .

(iii) Note also that, choosing  $\mathcal{E} = \mathcal{E}_{cc}$  the cocompact externology,  $H_{\mathcal{E}}^*(X) = H_c^*(X)$  is the classical compact supported cohomology of  $X$ . Moreover, see for instance [24, Prop. 3.6], whenever  $X$  is Hausdorff and locally compact,  $H_c^*(X)$  coincide with the reduced cohomology of the Alexandroff one-point compactification  $X^+$  of  $X$ , that is,

$$H_c^*(X) = \ker(H^*(X^+) \rightarrow H^*(\infty)) = \tilde{H}^*(X^+) \quad (2.0.1)$$

In order to show important properties of exterior cohomology, an analogous equation comparing the exterior cohomology of a given exterior space  $X$  with the reduced cohomology of the so called *Alexandroff exterior construction* of  $X$  will be considered. We explicitly recall this construction, studied in [15] to extend the classical result of Dold on partitions of unity [10, Thm. 6.1], to the proper category.

**Definition 2.4.** Given an exterior space  $(X, \mathcal{E})$  with topology  $\tau$ , the *Alexandroff exterior construction* of  $X$  is the based topological space  $X^\infty$  defined as the disjoint union of  $X$  with the based point  $\infty$ , and endowed with the topology  $\tau^\infty = \tau \cup \{E \cup \{\infty\}, E \in \mathcal{E}\}$ .

If  $\mathbf{Top}^*$  denotes the category of pointed spaces and maps, the above construction defines a functor

$$(-)^\infty: \mathbf{E} \longrightarrow \mathbf{Top}^*,$$

for which the following holds.

**Proposition 2.5.** *The functor  $(-)^\infty$  preserves:*

- (i) *Small limits and colimits. In particular  $(X \cup Y)^\infty = X^\infty \cup Y^\infty$  and  $(X \cap Y)^\infty = X^\infty \cap Y^\infty$  for any  $X, Y \in \mathbf{E}$ .*

- (ii) *Cylinders, that is,  $(X \bar{\times} I)^\infty \cong (X^\infty \times I)/\{\infty\} \times I$ , for any  $X \in \mathbf{E}$ . In particular,  $(-)^\infty$  preserves homotopies.*
- (iii) *Cofibrations and homotopy equivalences, i.e., for any exterior cofibration (resp. exterior homotopy equivalence)  $f: X \rightarrow Y$ , the map  $f^\infty: X^\infty \rightarrow Y^\infty$  is a classical pointed cofibration (resp. pointed homotopy equivalence).*
- (iv) *Homotopy colimits.*

*Proof.* For (i),(ii) and preservation of homotopy equivalences we refer to [15, §2,3]. Preservation of cofibrations and homotopy colimits are then immediate consequences.  $\square$

**Corollary 2.6.** *In particular,  $(-)^\infty$  induces a functor denoted in the same way*

$$(-)^\infty: \mathbf{HoCW}^{\mathbb{R}_+} \longrightarrow \mathbf{HoTop}^*.$$

$\square$

We also point out that, for any  $X \in \mathbf{HoCW}^{\mathbb{R}_+}$ ,  $X^\infty$  is Hausdorff and paracompact [15, Prop.14].

Next, we show that Equation 2.0.1 also holds in the exterior setting.

**Theorem 2.7.** *For any exterior space  $(X, \mathcal{E})$ ,*

$$H_{\mathcal{E}}^*(X) \cong \tilde{H}^*(X^\infty) = \ker(H^*(X^\infty) \rightarrow H^*(\infty)).$$

*Proof.* First, observe that  $X^\infty$  is an exterior space endowed with the externology  $\mathcal{E}^\infty$  given by the open neighborhoods of  $\infty$ , that is  $\mathcal{E}^\infty = \{E \cup \{\infty\}, E \in \mathcal{E}\}$ . Moreover, the inclusion  $X \hookrightarrow X^\infty$  is an exterior map. Hence, by the continuity of  $H^*$ , considering the triad  $(X^\infty, X, \emptyset)$ , and taking into account that  $X^\infty$  is the only closed exterior neighborhood of  $X$ ,

$$H_{\mathcal{E}}^*(X) = \varinjlim H^*(X, E) \cong \varinjlim H^*(X^\infty, B),$$

where  $B$  ranges over the family of closed exterior neighborhoods of  $X^\infty$  (that is,  $B$  is closed in  $X^\infty$  and contains an exterior set, also of  $X^\infty$ ). But this family is precisely that of all closed neighborhoods of the point  $\infty$  which is itself closed. Again by the continuity of  $H^*$ ,

$$\varinjlim H^*(X^\infty, B) \cong H^*(X^\infty, \infty)$$

and the theorem is proved.  $\square$

### 3 Brown representability of exterior cohomology

In the most general categorical framework, Brown representability for set-valued contravariant functors, defined on the homotopy category of a given closed model category, is now classical and well understood. See [9, 22, 23], or [21, Thm. 19] for a particularly explicit and precise statement. However, these results are not readily applicable to the exterior cohomology functor on  $\mathbf{E}_w^{\mathbb{R}+}$ . Indeed, on the one hand, it is not clear which family of objects is the one to choose so that it compactly generates the exterior category. On the other hand, as  $\mathbf{E}_w^{\mathbb{R}+}$  is not in general closed for limits or colimits, the required general form of the “wedge” and “Mayer-Vietoris” properties (see for instance G3 and G4 of [21, §3]) for the exterior cohomology functor on cofibrant objects may not be attained.

We follow the original approach of Brown in [6], with the same notation and language, to prove Theorem 0.1 of the Introduction in this section.

In what follows, an exterior based space  $(X, \mathbb{R}_+)$  will often be denoted simply by  $X$ .

We begin by proving all necessary properties on the homotopy category of (finite) based exterior CW-complexes.

**Lemma 3.1.**

- (i) *The category  $\mathbf{HoCW}^{\mathbb{R}+}$  (resp.  $\mathbf{HoCW}_f^{\mathbb{R}+}$ ) has arbitrary (resp. finite) coproducts.*
- (ii) *Any diagram in  $\mathbf{HoCW}^{\mathbb{R}+}$  or  $\mathbf{HoCW}_f^{\mathbb{R}+}$  of the form  $X_1 \xleftarrow{h_1} A \xrightarrow{h_2} X_2$  has a weak pushout.*
- (iii) *Any given direct system in  $\mathbf{HoCW}^{\mathbb{R}+}$ ,*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \dots ,$$

*admits a weak colimit  $(Y, \{g_n\}_{n \geq 1})$  for which the natural maps*

$$[Y, Z]^{\mathbb{R}+} \twoheadrightarrow \lim [X_n, Z]^{\mathbb{R}+} \quad \text{and} \quad \text{colim } [Z, X_n]^{\mathbb{R}+} \xrightarrow{\cong} [Z, Y]^{\mathbb{R}+}$$

*are, respectively, a surjection for every  $Z \in \mathbf{HoCW}^{\mathbb{R}+}$ , and a bijection for every  $Z \in \mathbf{HoCW}_f^{\mathbb{R}+}$ .*

*Proof.* (i) Given any family  $\{X_i, \}_{i \in I}$  of based exterior CW-complexes, the exterior (homotopy) pushout

$$\begin{array}{ccc} \coprod_{i \in I} (\mathbb{R}_+)_i & \longrightarrow & \mathbb{R}_+ \\ \downarrow & & \downarrow \\ \coprod_{i \in I} X_i & \longrightarrow & \bigvee_{i \in I}^{\mathbb{R}_+} X_i \end{array} \quad (3.0.2)$$

is easily checked to be their coproduct in  $\mathbf{HoCW}^{\mathbb{R}_+}$ . Observe that, if each  $(X_i, \mathbb{R}_+)$  and  $I$  are finite, then  $\bigvee_{i \in I}^{\mathbb{R}_+} X_i \in \mathbf{CW}_f^{\mathbb{R}_+}$ .

(ii) The homotopy pushout of  $X_1 \xleftarrow{h_1} A \xrightarrow{h_2} X_2$  is obviously a weak pushout. For it to lie within the category  $\mathbf{HoCW}^{\mathbb{R}_+}$  or  $\mathbf{HoCW}_f^{\mathbb{R}_+}$ , it is enough to choose exterior cellular representatives (see [14, Thms. 13,14]) of  $h_1$  and  $h_2$  and their factorization through exterior mapping cylinders.

(iii) Again, choosing exterior cellular representatives and factorizations through exterior mapping cylinders we may assume, without losing generality that each  $f_n$  of the direct diagram is a cellular cofibration. This way its colimit  $(Y, \{g_n\}_{n \geq 1})$  is a based exterior CW-complex in which each  $g_n: X_n \rightarrowtail Y$  is a cofibration. Surjectivity of the first map and injectivity of the second are obviously satisfied. For the onto character of the second map it is enough to apply [12, Prop. 4.1.21] or [16, Prop. 4.2] which, for convenience, we recall here: let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \dots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \dots$$

be a sequence of injective closed and e-closed exterior maps for which the points of  $X_k \setminus X_1$  are also closed and e-closed in  $X_k$ , for any  $k$ . As before, if we denote by  $(Y, \{g_n\}_{n \geq 1})$  the colimit of this sequence in the exterior category, then each exterior map  $h: Z \rightarrow Y$  from a Hausdorff, locally compact,  $\sigma$ -compact space with the cocompact externology, factors through  $X_n$  for  $n$  sufficiently large, i.e. there is an exterior map  $h_n: Z \rightarrow X_n$  such that  $g_n h_n = h$ .  $\square$

Next, we see that, on  $\mathbf{HoCW}^{\mathbb{R}_+}$ , the exterior cohomology functor satisfies the “wedge” and “Mayer-Vietoris” properties. For use in what follows, we remark that for any cohomology theory  $H^*$ , the reduced functor  $\tilde{H}^*$  on the based category satisfied these properties.

**Lemma 3.2.** *If  $\{X_i\}_{i \in I}$  is any family of based exterior CW-complexes, then there exists a canonical isomorphism*

$$H_{\mathcal{E}}^*(\bigvee_{i \in I}^{\mathbb{R}_+} X_i) \cong \prod_{i \in I} H_{\mathcal{E}}^*(X_i).$$

*Proof.* Applying the functor  $(-)^{\infty}$  to 3.0.2 we obtain, by Proposition 2.5, a (homotopy) pushout in the classical pointed category,

$$\begin{array}{ccc} \bigvee_{i \in I} (\mathbb{R}_+)_i^{\infty} & \longrightarrow & (\mathbb{R}_+)^{\infty} \\ \downarrow & & \downarrow \\ \bigvee_{i \in I} X_i^{\infty} & \longrightarrow & (\bigvee_{i \in I}^{\mathbb{R}_+} X_i)^{\infty}. \end{array}$$

Indeed, note that the inclusion  $(\coprod_{i \in I} (\mathbb{R}_+)_i)^{\infty} \hookrightarrow (\coprod_{i \in I} X_i)^{\infty}$  is homeomorphic to  $\bigvee_{i \in I} (\mathbb{R}_+)_i^{\infty} \hookrightarrow \bigvee_{i \in I} X_i^{\infty}$ . On the other hand, since the top row is trivially a (classical) homotopy equivalence, so is the bottom row. Applying this and Theorem 2.7 we get,

$$H_{\mathcal{E}}^*(\bigvee_{i \in I}^{\mathbb{R}_+} X_i) \cong \tilde{H}^*((\bigvee_{i \in I}^{\mathbb{R}_+} X_i)^{\infty}) \cong \tilde{H}^*(\bigvee_{i \in I} X_i^{\infty}) \cong \prod_{i \in I} \tilde{H}^*(X_i^{\infty}) \cong \prod_{i \in I} H_{\mathcal{E}}^*(X_i). \quad \square$$

**Lemma 3.3.** *If*

$$\begin{array}{ccc} A & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X \end{array}$$

*is an exterior homotopy pushout of exterior based CW-complexes, then the induced homomorphism*

$$H_{\mathcal{E}}^*(X) \longrightarrow H_{\mathcal{E}}^*(X_1) \times_{H_{\mathcal{E}}^*(A)} H_{\mathcal{E}}^*(X_2)$$

*is surjective*

*Proof.* An argument by factorizations through exterior mapping cylinders and exterior cellular approximations lets us assume, without losing generality, that the above is a pushout in  $\mathbf{HoCW}^{\mathbb{R}_+}$  where all maps are closed exterior cofibrations, and therefore  $A = X_1 \cap X_2$  and  $X = X_1 \cup X_2$ . Applying the functor  $(-)^{\infty}$ , see Proposition 2.5, we obtain another pushout with analogous properties in the based homotopy category

$$\begin{array}{ccc} A^{\infty} & \hookrightarrow & X_2^{\infty} \\ \downarrow & & \downarrow \\ X_1^{\infty} & \hookrightarrow & X^{\infty}. \end{array}$$

In other words,  $(X^\infty; X_1^\infty, X_2^\infty)$  is a classical excisive triad and the Mayer-Vietoris property for reduced cohomology asserts, in particular, that the following

$$\tilde{H}^*(X^\infty) \longrightarrow \tilde{H}^*(X_1^\infty) \oplus \tilde{H}^*(X_2^\infty) \longrightarrow \tilde{H}^*(A^\infty)$$

is exact. This, by Theorem 2.7, is equivalent to saying that

$$H_{\mathcal{E}}^*(X) \longrightarrow H_{\mathcal{E}}^*(X_1) \times_{H_{\mathcal{E}}^*(A)} H_{\mathcal{E}}^*(X_2)$$

is surjective.  $\square$

We are now able to prove the main results of the paper.

*Proof of Theorem 0.1.* Lemmas 3.2 and 3.3 above show that the pair  $(\mathbf{HoCW}^{\mathbb{R}_+}, \mathbf{HoCW}_f^{\mathbb{R}_+})$  and the functors  $H_{\mathcal{E}}^n$ ,  $n \geq 0$ , are respectively, a “homotopy category” and “homotopy functors” in the sense of [6, §2]. In our setting, the first part of Theorem 2.8 of [6] translates to the fact that, for each  $n \geq 0$ , there exists an exterior CW-complex  $K_n \in \mathbf{HoCW}^{\mathbb{R}_+}$  and a universal element  $u \in H_{\mathcal{E}}^n(K_n)$ , such that the natural natural transformation

$$T_u : [-, K_n]^{\mathbb{R}_+} \longrightarrow H_{\mathcal{E}}^n(-), \quad T_u[f] = H_{\mathcal{E}}^n(f)(u),$$

induces a bijection  $[X, K_n]^{\mathbb{R}_+} \cong H_{\mathcal{E}}^n(X)$  for any exterior finite CW-complex  $X \in \mathbf{HoCW}_f^{\mathbb{R}_+}$ . To prove, also applying directly Theorem 2.8 of [6], that  $K_n$  is unique up to based exterior homotopy, we see first that  $K_n$  is e-path connected. On the one hand, for the classical path-connectivity of  $K_n$ , consider the following general situation: let  $S_0^n$  denote the base ray  $\mathbb{R}_+$  with an  $n$ -sphere attached at 0, i.e, the pushout of  $\mathbb{R}_+ \leftarrow \{0\} \rightarrow S^n$ . If  $\theta$  denotes the obvious base ray of  $S_0^n$  and  $(X, \alpha)$  is any based exterior space, then there is an isomorphism,

$$[(S_0^n, \theta), (X, \alpha)]^{\mathbb{R}_+} = \pi_n(X, \alpha(0)).$$

Therefore

$$\pi_0(K_n) \cong [S_0^0, K_n]^{\mathbb{R}_+} \cong H_{\mathcal{E}}^n(S_0^0) \cong \tilde{H}^n((S_0^0)^+) \cong 0.$$

On the other hand,

$$\pi_0^B(K_n) \cong [\mathfrak{S}_B^0, K_n]^{\mathbb{R}_+} \cong H_{\mathcal{E}}^n(\mathfrak{S}_B^0) \cong \tilde{H}^n((\mathfrak{S}_B^0)^+) \cong 0.$$

We finish by remarking that the subcategory of  $\mathbf{HoCW}^{\mathbb{R}_+}$  consisting of e-path connected, exterior CW-complexes is compactly generated by

$\mathbf{HoCW}_f^{\mathbb{R}_+}$ . That is, a map  $f: X \rightarrow Y$  between e-path connected exterior CW-complexes is an exterior based homotopy equivalence if and only if  $f_*: [Z, X]^{\mathbb{R}_+} \rightarrow [Z, Y]^{\mathbb{R}_+}$  is a bijection for every  $Z \in \mathbf{CW}_f^{\mathbb{R}_+}$ . In fact, by the *exterior Whitehead theorem* [14, Thm. 12], as well as its classical version, e-path connected, based, exterior CW complexes are compactly generated simply by the finite based exterior CW-complexes  $S_\eta^n$  and  $S_0^n$ ,  $n \geq 1$ . For it, take into account that

$$[(S_\eta^n, \xi), (X, \alpha)]^{\mathbb{R}_+} \cong [(\mathfrak{S}^n, \eta), (X, \alpha)]^{\mathbb{N}} = \pi_n^{\mathcal{B}}(X, \alpha)$$

and the above isomorphism,

$$[(S_0^n, \theta), (X, \alpha)]^{\mathbb{R}_+} = \pi_n(X, \alpha(0))$$

for any based exterior space  $(X, \alpha)$ . This completes the proof.  $\square$

In the special case of considering classical cohomology with compact supports we obtain:

*Proof of Theorem 0.2.* Simply note that any countable, locally finite, finite dimensional relative CW-complex  $(X, \mathbb{R}_+)$  is an exterior finite based CW-complex endowed with the cocompact externology, see [18], [14, §2.1] or [16, §5.B]).  $\square$

We finish by characterizing, as in the classical case, the classifying space  $K_n$ . For it recall that, given an integer  $m \geq 1$  and a group  $G$  (abelian if  $m \geq 2$ ), we will denote by  $K_B(G, m)$  the *Eilenberg-MacLane exterior space*  $K_B(G, m)$  for *Brown-Grossman homotopy groups* [11], whose homotopy type in  $\mathbf{CW}^{\mathbb{R}_+}$  is unique due to the exterior Whitehead theorem [14, Thm. 12]. In what follows,  $H^*$  will denote either singular, Čech, Alexander-Spanier, or any other isomorphic cohomology theory on HLC spaces. In fact, we only need  $H_{\mathcal{E}}^*$  to be isomorphic to any of the above on  $S_\eta^m$ , for any  $m \geq 1$ . In this context we are also allowed to consider a coefficient ring  $R$  which is fixed henceforth.  $R^\infty$  will denote the product of countably infinitely many copies of  $R$ .

**Theorem 3.4.** *For any  $n \geq 1$ , the classifying space  $K_n$  is exterior homotopy equivalent to  $K_B(R^\infty, n)$ .*

*Proof.* In view of Theorem 0.1, we have a bijection,

$$\pi_m^{\mathcal{B}}(K_n, \alpha) = [(S_\eta^m, \xi), K_n]^{\mathbb{R}_+} \cong H_{\mathcal{E}}^n(S_\eta^m) \cong H_c^n(S_\eta^m) \cong \begin{cases} R^\infty, & n = m; \\ 0, & n \neq m. \end{cases}$$

To prove that this is in fact an isomorphism it is enough to check that exterior cohomology respects the addition in  $\pi_m^{\mathcal{B}}(K_n, \alpha)$ , that is,  $H_{\mathcal{E}}^n(f + g) = H_{\mathcal{E}}^n(f) + H_{\mathcal{E}}^n(g)$  for any  $f, g \in \pi_n^{\mathcal{B}}(K_n, \alpha)$ . More generally, for every based exterior CW-complex  $X$  and any pair of exterior maps  $f, g: S_{\eta}^m \rightarrow X$  representing elements of  $\pi_m^{\mathcal{B}}(X)$  we show that

$$H_{\mathcal{E}}^n(f + g) = H_{\mathcal{E}}^n(f) + H_{\mathcal{E}}^n(g): H_{\mathcal{E}}^n(X) \longrightarrow H_{\mathcal{E}}^n(S_{\eta}^m).$$

Here, abusing of notation, we do not distinguish maps from the homotopy classes which they represent. Due to the representability of the exterior cohomology, the above is equivalent to seeing that

$$(f + g)^* = f^* + g^*: [X, K_n]^{\mathbb{R}_+} \longrightarrow [S_{\eta}^m, K_n]^{\mathbb{R}_+},$$

where  $(-)^*$  denotes composition on the right.

For it, consider the isomorphism

$$w_X: \pi_m^{\mathcal{B}}(X) \cong [S_{\eta}^m, X]^{\mathbb{R}_+} \cong [S^m, X^{\mathbb{N}}] = \pi_m(X^{\mathbb{N}})$$

and observe that its naturality provides, for every  $h \in [X, K_n]^{\mathbb{R}_+}$ , a commutative square

$$\begin{array}{ccc} \pi_m^{\mathcal{B}}(X) & \xrightarrow{h_*} & \pi_m^{\mathcal{B}}(X) \\ w_X \downarrow \cong & & \cong \downarrow w_{K_n} \\ \pi_m(X^{\mathbb{N}}) & \xrightarrow{(h^{\mathbb{N}})_*} & \pi_m(K_n^{\mathbb{N}}). \end{array}$$

Here  $(-)_*$  denotes composition on the left. Therefore

$$w_{K_n}(h(f + g)) = w_{K_n}h_*(f + g) = (h^{\mathbb{N}})_*w_X(f + g) = w_{K_n}(hf + hg).$$

and  $h(f + g) = hf + hg$ , as required.  $\square$

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